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A Reducibility Theorem for Holomorphic Quasi-Periodic Linear Systems Via an Implicit Function Theorem

ALISTAIR GRAY

Department of Pure Mathematics, La Trobe University, Bundoora 3083, Australia

Submitted by Alex McNabb

1. INTRODUCTION

In [1], a single theorem that is capable of dealing with many small divisor problems was given together with two applications, these being a proof of the well-known Moser twist theorem and a new generalization of the Siegel center theorem. This theorem was based on the work of Sternberg [4]. Both applications are essentially conjugacy problems as were all of the applications considered by Sternberg in [4]. In this paper, we use the same theorem (with one slight modification) to deal with a problem, that is apparently not a conjugacy problem.

The application here is to a reducibility theorem for holomorphic quasi-periodic linear systems of differential equations, which was first proved by Mitropol'skiĭ and Samoĭlenko [2], and the estimates later greatly improved by O'Brien [3]. The proof here turns out to be a straighforward verification of the hypotheses of the accelerated convergence theorem of [1].

The iterative process used in the abstract theorem gives rise to a Cauchy sequence of approximations to the solution. The key step here is to establish at each stage of the iteration an estimate of the error. This problem is solved by the inductive argument given towards the end of Section 5 of this paper. Previous authors [2, 3] established a Cauchy sequence, but used a rather indirect method to show that the limit satisfied the appropriate equation.

Before stating the reducibility theorem we introduce some notation.

2. PRELIMINARY DEFINITIONS AND NOTATION

Let Z, C, and R denote the sets of integers, complex numbers, and real numbers respectively. $\mathbb{C}^{q\times q} = \{q \times q$ matrices over the complex field$\}$. Let $A \subset \mathbb{C}^r$, and define $\mathscr{P}\mathscr{F}(A, \mathbb{C}^{q\times q}) = \{$functions holomorphic on domain A

having codomain $\mathbb{C}^{q\times q}$ with period 2π in each variable}. We define subsets of $\mathbb{C}^r$ for $h > 0$ as follows

$$S^r[h] = \{x \in \mathbb{C}^r: |\operatorname{im} x^i| < h, i = 1, 2,..., r\}$$

$$\bar{S}^r[h] = \{x \in \mathbb{C}^r: |\operatorname{im} x^i| \leqslant h, i = 1, 2,..., r\}.$$

Let $\mathscr{F}(\mathbb{R}, \mathbb{R}^q)$, $\mathscr{F}(\mathbb{R}, \mathbb{R}^r)$ be the sets of once differentiable functions with domain $\mathbb{R}$ and codomains $\mathbb{R}^q$ and $\mathbb{R}^r$, respectively.

Let M, p, c, w, and u be positive constants. We define some function spaces and norms by

$$\mathscr{B}_0 = \mathscr{P}\mathscr{F}(S^r[3p], \mathbb{C}^{q\times q}),$$

$$\mathscr{Y} = \mathscr{P}\mathscr{F}(\bar{S}^r[2p], \mathbb{C}^{q\times q}),$$

$$\| \beta \|_{\mathscr{B}_0} = \max_j \sum_i \sup_{x\in S^r[3p]} |\beta^{ij}(x)|,$$

and

$$\| \mu \|_{\mathscr{Y}} = \max_j \sum_i \sup_{x\in \bar{S}^r[2p]} |\mu^{ij}(x)|,$$

where $1 \leqslant i, j \leqslant q$.

The vector norm "$\| \;\|$" on the space $\mathbb{C}^r$ is defined by $\| z \| = \sum_{i=1}^r | z^i |$ and "(z_1, z_2)" is the usual inner product of two vectors z_1 and z_2 in $\mathbb{C}^r$. Ordinary matrix multiplication and the operation of taking the pointwise product of matrix functions will be denoted by "$\cdot$". By "D" we denote Fréchet differentiation and by "D_i" we denote Fréchet differentiation with respect to the ith variable.

3. The Reducibility Theorem for Holomorphic Quasi-periodic Linear Systems

THEOREM. *Let $\alpha_0 \in \mathscr{B}_0$ and assume that whenever $x \in \mathbb{R}^r$ we have $\alpha_0(x) \in \mathbb{R}^{q\times q}$. Assume A_0 is a real constant diagonal $q \times q$ matrix all of whose eigenvalues differ by at least u. (Recall $u > 0$). Now suppose $w > r$, $d \in \mathbb{R}^r$ with $|(d, z)| \geqslant M \| z \|^{-w}$ for each $z \in \mathbb{Z}^r\backslash\{0\}^r$. Then provided $\| \alpha_0 \|_{\mathscr{B}_0}$ is small enough (how small depends only on M, p, r, w and u) the quasi-periodic linear system*

$$\eta_0'(t) = (A_0 + \alpha_0(td)) \cdot \eta_0(t), \qquad \eta_0 \in \mathscr{F}(\mathbb{R}, \mathbb{R}^q)$$

has a fundamental matrix of the form

$$\chi(t) = (I + \zeta(td)) \cdot e^{tA},$$

where I denotes the identity matrix, $\zeta \in \mathscr{V}$ is bounded, and Λ is a real constant diagonal $q \times q$ matrix all of whose eigenvalues differ by at least $\frac{1}{2}u$.

As in the other applications of the accelerated convergence theorem [1], we restate this problem as the functional equation for ζ assuming that α_0 is fixed:

$$G(\zeta, \alpha_0) = 0.$$

Next we quote, without including proofs, the accelerated convergence theorem as it applies to this problem.

4. The Accelerated Convergence Theorem

THEOREM. *Let $\mathscr{X}$ be a neighbourhood of the origin of a Banach space $\mathscr{V}$, $\mathscr{A}_0$ a neighborhood of the origin of the normed vector space $\mathscr{B}_0$, and let $\mathscr{B}$ be a Banach space. Assume the map*

$$G\colon \mathscr{X} \times \mathscr{A}_0 \to \mathscr{B} \tag{1}$$

is continuous.

The existence of the sequences of spaces and maps that are listed below is assumed and $\| \ \|_{\mathscr{U}}$ will be used to denote the norm on a normed space $\mathscr{U}$ while g, k, s, and t are nonnegative constants that satisfy

$$1 < t < 2, \tag{2}$$

$$(\log 2 + k)/(2 - t) < s, \tag{3}$$

$$(\log 6 + gt(t - 1))/(t - 1) < s. \tag{4}$$

$(\mathscr{B}_n)_{n=0}^{\infty}$ and $(\mathscr{V}_n)_{n=0}^{\infty}$ are sequences of normed vector spaces with neighborhoods of the origin as follows

$$\mathscr{A}_n = \{\alpha \in \mathscr{B}_n \colon \| \alpha \|_{\mathscr{B}_n} < e^{-st^n}\}, \tag{5}$$

$$\mathscr{Y}_n = \{\xi \in \mathscr{V}_n \colon \| \xi \|_{\mathscr{V}_n} < e^{(g-s)t^n}\}, \tag{6}$$

$$\mathscr{X}_n = \{\zeta \in \mathscr{V}_n \colon \| \zeta \|_{\mathscr{V}_n} < e^{g-s}(2 - 2^{-(n+1)})\}, \tag{7}$$

while $\mathscr{X}$ is the subset of $\mathscr{V}$:

$$\mathscr{X} = \{\zeta \in \mathscr{V} \colon \| \zeta \|_{\mathscr{V}} \leqslant 2e^{g-s}\}. \tag{8}$$

$(G_{n+1})_{n=0}^{\infty}$ is a sequence of twice differentiable maps and

$$G_{n+1}\colon \mathscr{Y}_n \times \mathscr{A}_n \to \mathscr{B}_{n+1}, \tag{9}$$

$$G_{n+1}(0, 0) = 0, \tag{10}$$

and whenever (μ, ν) *and* $(\xi, \alpha) \in \mathscr{Y}_n \times \mathscr{A}_n$

$$\| D^2G_{n+1}(\mu, \nu)((\xi, \alpha), (\xi, \alpha))\|_{\mathscr{B}_{n+1}} < e^{(k-2s)t^n}. \tag{11}$$

$(L_n)_{n=0}^{\infty}$ *is a sequence of maps such that*

$$L_n : \mathscr{A}_n \to \mathscr{Y}_n, \tag{12}$$

$$\| DG_{n+1}(0, 0)(L_n(\alpha), \alpha)\|_{\mathscr{B}_{n+1}} < e^{(k-2s)t^n}, \tag{13}$$

whenever $\alpha \in \mathscr{A}_n$.

$(I_n)_{n=0}^{\infty}$ *and* $(J_n)_{n=0}^{\infty}$ *are sequences of linear norm-decreasing maps such that*

$$I_n : \mathscr{V}_n \to \mathscr{V}, \qquad J_n : \mathscr{V}_n \to \mathscr{V}_{n+1}, \qquad I_{n+1} \circ J_n = I_n. \tag{14}$$

The sequence of maps $(\oplus_{n+1})_{n=0}^{\infty}$ *has the properties*

$$\oplus_{n+1} : \mathscr{X}_n \times \mathscr{Y}_{n+1} \to \mathscr{V}_{n+1} : (\zeta, \xi) \mapsto \zeta \oplus_{n+1} \xi, \tag{15}$$

$$\left\| \zeta \oplus_{n+1} \xi - J_n(\zeta) - \xi \right\|_{\mathscr{V}_{n+1}} \leqslant \| \zeta \|_{\mathscr{V}_n} \| \xi \|_{\mathscr{V}_{n+1}}, \tag{16}$$

whenever $\zeta \in \mathscr{X}_n$ *and* $\xi \in \mathscr{Y}_{n+1}$.

It was shown in [1] *that the sequences* $(\alpha_n)_{n=0}^{\infty}$, $(\xi_n)_{n=0}^{\infty}$, *and* $(\zeta_n)_{n=0}^{\infty}$, *which were defined recursively by the relations*

$$\xi_n = L_n(\alpha_n), \qquad \alpha_{n+1} = G_{n+1}(\xi_n, \alpha_n), \tag{17}$$

$$\zeta_0 = \xi_0, \qquad \zeta_{n+1} = \zeta_n \oplus_{n+1} \xi_{n+1}, \tag{18}$$

(*assuming* α_0 *is fixed*), *have the properties* $\xi_n \in \mathscr{Y}_n$, $\alpha_n \in \mathscr{A}_n$, *and* $\zeta_n \in \mathscr{X}_n$. *The following inequality is assumed valid for each n.*

$$\| G(I_n(\zeta_n), \alpha_0)\|_{\mathscr{B}} \leqslant e^{kt^{n+1}}(\| \alpha_{n+1} \|_{\mathscr{B}_{n+1}} + e^{-st^{n+1}}). \tag{19}$$

Then for each $\alpha_0 \in \mathscr{A}_0$, *there is a* $\zeta \in \mathscr{X}$ *such that*

$$G(\zeta, \alpha_0) = 0. \tag{20}$$

Remark. The above theorem, except for (19), can be obtained from the accelerated convergence theorem in [1] by putting $v = m = 0$, $l = 2$, $\||\ \||_{\mathscr{B}_n} = \|\ \|_{\mathscr{B}_n}$, and $\mathscr{H}_n = \mathscr{B}_n$ (see [1] for details). To obtain (19) here, we alter inequation (25) of [1] to

$$\| G(I_n(\zeta_n), \alpha_0)\|_{\mathscr{B}} \leqslant e^{(k+vs)t^{n+1}}(\| \alpha_{n+1} \|_{\mathscr{B}_{n+1}} + e^{-st^{n+1}}).$$

This change does not invalidate the proofs in [1].

We now verify that the hypotheses of this theorem can be satisfied in the case of the reduciblity theorem quoted above. Note that here, as in the other applications "s large" is effectively "$\| \alpha_0 \|_{\mathscr{B}_0}$ small."

5. Verification of Hypotheses of the Accelerated Convergence Theorem

The Map G

Given the linear differential equation of the reducibility theorem

$$\eta_0'(t) = (\Lambda_0 + \alpha_0(td)) \cdot \eta_0(t),$$

we wish to find a matrix $\zeta(td)$ such that

$$\eta_0(t) = (I + \zeta(td)) \cdot \eta(t),$$

and

$$\eta'(t) = \Lambda \cdot \eta(t),$$

where Λ is a constant diagonal matrix whose eigenvalues differ by at least $\frac{1}{2}u$.

This problem can be written as the functional equation (20) where G is given by

$$G(\xi, \alpha_0) = (\mathbf{I} + \zeta)^{-1} \cdot ((\mathbf{\Lambda}_0 + \alpha_0) \cdot (\mathbf{I} + \zeta) - P_d(\zeta)) - \mathbf{\Lambda}, \tag{21}$$

P_d by

$$P_d(\zeta)(\cdot) = \sum_{i=1}^{r} D_i\zeta(\cdot)(d^i),$$

and $\mathbf{I}(x) = I$, $\mathbf{\Lambda}_0(x) = \Lambda_0$, $\mathbf{\Lambda}(x) = \Lambda$, for each $x \in \mathbb{C}^r$.

The constant matrix Λ will be determined by the sequence $(\alpha_n)_{n=0}^{\infty}$ and Λ_0. The proof of (1) is postponed until Λ has been defined.

Domains, Estimates, and Function Spaces

As in the Moser twist theorem, the construction of the various sequences of function spaces is motivated by the need to obtain bounds for the derivatives in terms of bounds of the function at each step of the iteration. In this case we use the result

$$\sup_{x \in S^r[h_1]} | D_i\mu(x) \cdot (y)| \leqslant \frac{2}{(h - h_1)^2} \sup_{x \in S^r[h]} | \mu(x)| , \tag{22}$$

where $\mu \in \mathscr{P}\mathscr{F}(S^r[h], \mathbb{C})$, $0 < h_1 < h$, and $i \in \{1, 2,..., r\}$.

We now define some domains and function spaces

$$\begin{aligned} B_n &= S^r[2p(1+2^{-(n+1)})], & \mathscr{B}_n &= \mathscr{P}\mathscr{F}(B_n, \mathbb{C}^{q\times q}),\\ V_n &= S^r[2p(1+2^{-(n+1)})], & \mathscr{V}_n &= \mathscr{P}\mathscr{F}(V_n, \mathbb{C}^{q\times q}),\\ B &= S^r[p], & \mathscr{B} &= \mathscr{P}\mathscr{F}(B, \mathbb{C}^{q\times q}),\\ V &= \bar{S}^r[2p], & \mathscr{V} &= \mathscr{P}\mathscr{F}(V, \mathbb{C}^{q\times q}). \end{aligned}$$

Norms on each function space are defined thus

$$\| \mu \|_{\mathscr{U}} = \max_j \sum_{i=1}^{q} \sup_{x\in U} | \mu^{ij}(x)|,$$

where $\mu(x) = (\mu^{ij}(x))$ and $\operatorname{dom}(\mu \in \mathscr{U}) = U$.

The Map G_{n+1}

First we define the set $\mathscr{D}_h$ for $h > 0$.

$\mathscr{D}_h = \{$real constant diagonal $q \times q$ matrices whose eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_q$, satisfy $\min_{i\neq j} | \lambda_i - \lambda_j | > h\}$.

Note that $\Lambda_0 \in \mathscr{D}_u$. The sequence $(\Lambda)_{n=0}^{\infty}$ is defined recursively by the relation

$$\Lambda_{n+1} = \Lambda_n + \bar{\alpha}_n, \tag{23}$$

where $\bar{\alpha}_n = \operatorname{diag}((2\pi)^{-r} \int_T \alpha_n)$, with $T = [0, 2\pi]^r$.

Clearly then,

$$\Lambda_n \in \mathscr{D}_v, \qquad \text{where} \qquad v = u - 2\sum_{k=0}^{n} e^{-st^k},$$

and for s large enough, there is a $\Lambda \in \mathscr{D}_{(1/2)u}$ such that

$$\Lambda = \lim_{n\to\infty} \Lambda_n.$$

G_{n-1} is then defined by putting, for $\xi \in \mathscr{Y}_n$ and $\alpha \in \mathscr{A}_n$,

$$G_{n+1}(\xi, \alpha) = (\mathbf{I} + \xi)^{-1} \cdot (\mathbf{\Lambda}_n + \mathbf{\Lambda}_n \cdot \xi + \alpha + \alpha \cdot \xi - P_d(\xi)) - (\mathbf{\Lambda}_n + \bar{\boldsymbol{\alpha}}), \tag{24}$$

where $\bar{\boldsymbol{\alpha}}(x) = \bar{\alpha}$, for all $x \in \mathbb{C}^r$.

Now since for $\| \xi \|_{\mathscr{V}_n} < \frac{1}{2}$ (s large enough will ensure this for all n) we have

$$(\mathbf{I} + \xi)^{-1} = \mathbf{I} - \xi + \xi^2 - \xi^3 + \cdots. \tag{25}$$

The above and the triangle inequality gives

$$\|(\mathbf{I} + \xi)^{-1}\|_{\mathscr{V}_n} < 2,$$

and this is sufficient to establish (9) since ξ and α are clearly holomorphic and bounded on B_{n+1}. A similar argument establishes (1).

The First and Second Fréchet Derivatives of G_{n+1}

Differentiating G_{n+1} twice by using the definition of Fréchet differentiation and an elementary "first principles" technique leads to the expressions

$$\begin{aligned} DG_{n+1}(\zeta, \beta)(\xi, \alpha) = (\mathbf{I} + \zeta)^{-1} \cdot (\mathbf{\Lambda}_n \cdot \xi + \alpha + \beta \cdot \xi + \alpha \cdot \zeta \\ - P_d(\xi) - \xi \cdot (G_{n+1}(\zeta, \beta) + \mathbf{\Lambda}_n + \bar{\boldsymbol{\beta}})) - \bar{\boldsymbol{\alpha}}, \quad (26) \end{aligned}$$

$$\begin{aligned} &D^2G_{n+1}(\zeta, \beta)((\xi, \alpha), (\xi, \alpha)) \\ &\quad = 2(\mathbf{I} + \zeta)^{-1} \cdot (\alpha \cdot \xi - \xi \cdot (DG_{n+1}(\zeta, \beta)(\xi, \alpha) + \bar{\boldsymbol{\alpha}})). \qquad (27) \end{aligned}$$

The same argument as used in proving (9) is used to show that these two expressions are holomorphic on B_{n+1} whenever (ξ, α) and $(\zeta, \beta) \in \mathscr{Y}_n \times \mathscr{A}_n$. The estimates that establish (11) are easily obtainable from (27).

The Map L_n

From (26), putting $\zeta = \beta = 0$ we obtain

$$DG_{n+1}(0, 0) \cdot (\xi, \alpha) = \mathbf{\Lambda}_n \cdot \xi - \xi \cdot \mathbf{\Lambda}_n - P_d(\xi) + \alpha - \bar{\boldsymbol{\alpha}}.$$

O'Brien showed that for each $\alpha \in \mathscr{A}_n$, there is a unique $\xi \in \mathscr{V}_n$ such that

$$\xi \cdot \mathbf{\Lambda}_n - \mathbf{\Lambda}_n \cdot \xi + P_d(\xi) = \alpha - \bar{\boldsymbol{\alpha}},$$

and

$$\| \xi \|_{\mathscr{V}_n} \leqslant \left(\left(u - 2 \sum_{k=0}^{n} w^{-st^n}\right)^{-1} + \frac{C}{M} (p^{-1}\, 2^{(n+2))w}\right) \| \alpha \|_{\mathscr{B}_n},$$

where C depends only on r and w.

Clearly, there exists a g (which depends only on u, p, w, M, and t because

$$u - 2 \sum_{k=0}^{n} e^{-st^n} > \tfrac{1}{2} u,$$

such that

$$\left(u - 2 \sum_{k=0}^{n} e^{-st^n}\right)^{-1} + \frac{C}{M} (p^{-1}\, 2^{+(n+2))w} < e^{gt^n},$$

for all $n = 0, 1, 2, \ldots$, and so (12) is established as is (13).

The Maps I_n, J_n and $\oplus_{n+1}$

We define I_n as the map that restricts the domain of the (matrix) functions in $\mathscr{X}_n$ (the functions holomorphic and bounded on Γ_n) to Γ. The map J_n restricts the domain of functions in $\mathscr{X}_n$ to Γ_{n+1}. Hypothesis (14) is clearly satisfied by these definitions.

The map $\oplus_{n+1}$ is defined by putting for $\zeta \in \mathscr{X}_n$ and $\xi \in \mathscr{Y}_{n+1}$

$$\begin{aligned} \zeta \oplus_{n+1} \xi &= (\mathbf{I} + J_n(\zeta)) \cdot (\mathbf{I} + \xi) - \mathbf{I} \\ &= J_n(\zeta) + \xi + J_n(\zeta) \cdot \xi, \end{aligned} \tag{28}$$

which satisfies (15).

Thus

$$\begin{aligned} \| \zeta \oplus_{n+1} \xi - J_n(\zeta) - \xi \|_{\mathscr{Y}_{n+1}} &= \| J_n(\zeta) \cdot \xi \|_{\mathscr{Y}_{n+1}} \\ &\leqslant \| J_n(\zeta) \|_{\mathscr{Y}_{n+1}} \| \xi \|_{\mathscr{Y}_{n+1}} \\ &\leqslant \| \zeta \|_{\mathscr{X}_n} \| \xi \|_{\mathscr{Y}_{n+1}}, \end{aligned}$$

as required by (16).

The Map G and Inequality (19)

We first prove by induction that

$$(\mathbf{I} + \zeta_n)^{-1} \cdot ((\mathbf{\Lambda}_0 + \alpha_0) \cdot (\mathbf{I} + \zeta_n) - P_d(\zeta_n)) - \mathbf{\Lambda}_{n+1} = \alpha_{n+1} \tag{29}$$

over the domain B_{n+1}, where α_{n+1} is given by (17).

Since $\zeta_0 = \xi_0$ this is true identically for $n = 0$. Note that by (28), $\mathbf{I} + \zeta_{n+1} = (\mathbf{I} + \zeta_n) \cdot (\mathbf{I} + \xi_{n+1})$, and that P_d is a linear map such that

$$P_d(\mathbf{I}) = 0, \qquad P_d(\mu \cdot \eta) = \mu \cdot P_d(\eta) + P_d(\mu) \cdot \eta.$$

Consider now

$$\begin{aligned} &(\mathbf{I} + \zeta_{n+1})^{-1} \cdot ((\mathbf{\Lambda}_0 + \alpha_0) \cdot (\mathbf{I} + \zeta_{n+1}) - P_d(\zeta_{n+1})) \\ &\quad = (\mathbf{I} + \xi_{n+1})^{-1} \cdot (\mathbf{I} + \zeta_n)^{-1} \cdot [(\mathbf{\Lambda}_0 + \alpha_0) \cdot (\mathbf{I} + \zeta_n) \cdot (\mathbf{I} + \xi_{n+1}) \\ &\qquad\qquad - P_d((\mathbf{I} + \zeta_n) \cdot (\mathbf{I} + \xi_{n+1}) - \mathbf{I})] \\ &\quad = (\mathbf{I} + \xi_{n+1})^{-1} \cdot \{(\mathbf{I} + \zeta_n)^{-1} \cdot [(\mathbf{\Lambda}_0 + \alpha_0) \cdot (\mathbf{I} + \zeta_n) - P_d(\zeta_n)] \\ &\qquad\qquad \cdot (\mathbf{I} + \xi_{n+1}) - P_d(\xi_{n+1})\} \\ &\quad = (\mathbf{I} + \xi_{n+1})^{-1} \cdot \{(\mathbf{\Lambda}_{n+1} + \alpha_{n+1}) \cdot (\mathbf{I} + \xi_{n+1}) - P_d(\xi_{n+1})\}, \qquad \text{by (29)}, \\ &\quad = \alpha_{n+2} + \mathbf{\Lambda}_{n+2}, \qquad \text{by (17) and (24)}, \end{aligned}$$

which completes the induction.

Thus, for all $x \in B$ we have (since $B \subset B_{n+1}$) by (29), (23), and (24).

$$(G_{n+1}(\xi_n, \alpha_n) + \mathbf{\Lambda}_{n+1})(x) = (G(I_n(\zeta_n), \alpha_0) + \mathbf{\Lambda})(x)$$

and so

$$\| G(J_n(\zeta_n), \alpha_0)\|_{,s} = \max_j \sum_{i=1}^{q} \sup_{x\in B} | \alpha_{n+1}^{ij}(x) + (\Lambda^{ij} - \Lambda_{n+1}^{ij})|$$

$$\leqslant \| \alpha_{n+1} \|_{\mathscr{B}_{n+1}} + 2 \sum_{k=n+2}^{\infty} e^{-st^k}$$

$$\leqslant \| \alpha_{n+1} \|_{\mathscr{B}_{n+1}} + 4e^{-st^{n+1}},$$

which establishes (19) provided $e^k \geqslant 4$ and s is large enough to ensure that

$$e^{-st^{n+1}} \leqslant \tfrac{1}{2}e^{-st^n}.$$

6. Discussion

The main difference in approach here is that we employ Taylor estimates for $G_{n+1}(L_n(\alpha_n), \alpha_n)$ at each step instead of back-substitution followed by direct estimation as in [2] and [3]. The latter will give sharper estimates. The accelerated convergence theorem could be modified to take this into account by deleting (10) and (11) and replacing (13) by

$$\| G_{n+1}(L_n(\alpha), \alpha)\|_{\mathscr{B}_{n+1}} < e^{(k-2s)t^n}, \qquad \text{whenever} \qquad \alpha \in \mathscr{A}_n .$$

However, in this form the theorem would not be applicable to the Moser twist theorem.

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